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AN INVARIANT SOLUTION OF GAS DYNAMICS EQUATIONS

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The problem of an invariant solution of gas dynamics equations is considered. It was shown in [1] that in the case of an arbitrary equation of state, these equations admit the G_{11} group. Below, we study an invariant solution for a series of three-parameter subgroups from G_{11} containing two arbitrary constants (subalgebra 3.15 from Table 6 [1]) that exert an influence on the solution form. A nonsingular invariant solution exists when the constants do not vanish simultaneously. The solution is expressed by finite formulas. In the case where both constants are equal to zero, one can obtain a partially invariant solution with invariance defects 1 or 2. The study of the partially invariant solution of defect 1 and rank 2 shows that it belongs to regular solutions [2] and is reduced to an invariant solution with respect to the two-parameter group involved in the initial three-parameter group. This solution is described by a closed system of equations with two independent variables. A group classification with respect to the equation of state is performed for this system.

1. Preliminary Remarks. We consider the equations of gas dynamics

$$D\mathbf{u} + \rho^{-1}\nabla p = 0, \quad D\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad DS = 0, \quad p = f(\rho, S). \tag{1.1}$$

Here $D = \partial_t + \mathbf{u}\cdot\nabla$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\mathbf{u} = (u, v, w)$ is the velocity vector, ρ is the density, p is the pressure, and S is the entropy. All functions depend on time t and on the coordinates $\mathbf{x} = (x, y, z)$; f is a given function.

It is known (for instance, from [1]) that in the case of an arbitrary f, system (1.1) admits the eleven-parameter transformation group G_{11} of the base space $R^9(t, \mathbf{x}, \mathbf{u}, \rho, S)$. In this paper, we study an invariant solution constructed for a series of three-parameter subgroups $H(\alpha, \beta) \subset G_{11}$ with the Lie algebra generated by the operators

$$H_1 = \partial_z + t\partial_y + \partial_v, \quad H_2 = \partial_y - t\partial_z - \partial_w, \quad H_3 = \alpha\partial_x + \beta(t\partial_x + \partial_u) + y\partial_z - z\partial_y + v\partial_w - w\partial_v, \tag{1.2}$$

where α and β are arbitrary material parameters; note that subalgebras that are dissimilar with respect to the action of the groups of internal automorphisms G_{11} correspond to different values of α and β. Therefore, the solutions corresponding to different α and β differ significantly, i.e., cannot be transformed into one another by means of transformations from G_{11}.

It should be noted that the factor-system E/H always admits the normalizer H in G, i.e., the transformations admitted by the factor-system are partially known. In this problem, the Lie algebra generated by the operators

$$Y_1 = \partial_x, \quad Y_2 = t\partial_x + \partial_u, \quad Y_3 = \partial_z + t\partial_y + \partial_v, \quad Y_4 = \partial_y - t\partial_z - \partial_w,$$
$$Y_5 = y\partial_z - z\partial_y + v\partial_w - w\partial_v$$

corresponds to the normalizer $H(\alpha, \beta)$ in G_{11}. These transformations make it possible to simplify the form of the exact solution.

2. Conditions of Existence of an Invariant *H*-Solution. One should primarily verify the existence conditions of a nonsingular invariant H-solution. For this purpose, we proceed as follows. Denote

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$X = (t, x, y, z)$ and $Y = (u, v, w, \rho, S)$. A tangential mapping $\boldsymbol{\zeta}$ of the group $H(\alpha, \beta)$ is determined from Eqs. (1.2). Divide the general operator $\boldsymbol{\zeta} \cdot \boldsymbol{\partial}$ of algebra (1.2) into parts so that

$$\boldsymbol{\zeta} \cdot \boldsymbol{\partial} = \boldsymbol{\xi} \cdot \boldsymbol{\partial}_X + \boldsymbol{\eta} \cdot \boldsymbol{\partial}_Y.$$

Let $r_*(\boldsymbol{\zeta})$ be the general rank of the tangential mapping $\boldsymbol{\zeta}$. Then, for Eqs. (1.1) to have a nonsingular invariant H-solution, the relations

$$r_*(\boldsymbol{\xi}) = r_*(\boldsymbol{\zeta}) \leqslant n \tag{2.1}$$

should hold.

In our case, $n = 4$ and $r_*(\boldsymbol{\zeta}) = 3$, while

$$r_*(\boldsymbol{\xi}) = \begin{cases} 3, & \alpha^2 + \beta^2 \neq 0, \\ 2, & \alpha^2 + \beta^2 = 0. \end{cases}$$

Thus, condition (2.1) is satisfied only when $\alpha^2 + \beta^2 \neq 0$, i.e., when $\alpha^2 + \beta^2 = 0$, it is impossible to construct a nonsingular invariant H-solution.

3. The Case $\boldsymbol{\alpha^2 + \beta^2 \neq 0}$. We construct an invariant solution with respect to the group $H(\alpha, \beta)$, whose solution algorithm is well known from [3]. A universal invariant of the group $H(\alpha, \beta)$ can be selected as follows:

$$J = (t,\ \beta x - (\alpha + \beta t)u,\ U,\ x - (\alpha + \beta t)\,\Phi,\ \rho, S).$$

Here the new functions U and Φ are determined from the relations $y - tv + w = U \cos \Phi$ and $z - v - tw = U \sin \Phi$.

The rank of the solution is $\sigma = n - r_*(\zeta) = 1$; therefore, the invariant $H(\alpha, \beta)$-solution is represented as

$$\begin{gathered} u = \frac{\beta x - b(t)}{\alpha + \beta t}, \quad v = \frac{ty + z - U(t)(t \cos \Phi + \sin \Phi)}{t^2 + 1}, \\ w = \frac{tz - y - U(t)(t \sin \Phi - \cos \Phi)}{t^2 + 1}, \quad S = S(t), \quad \rho = \rho(t), \ \Phi = \frac{x - d(t)}{\alpha + \beta t}. \end{gathered} \tag{3.1}$$

One can see from (3.1) that the density in the required solution depends only on time, i.e., it describes a particular case of barochronic motions [4]. Substituting the solution representation into system (1.1), we obtain the factor-system E/H of ordinary differential equations:

$$b_t = 0, \quad U_t = 0, \quad d_t = \frac{\beta d - b}{\alpha + \beta t}, \quad \rho_t + \rho\Big(\frac{\beta}{\alpha + \beta t} + \frac{2t}{t^2 + 1}\Big) = 0, \quad S_t = 0. \tag{3.2}$$

Solving system (3.2) and using the representation (3.1), we find an explicit form of the invariant solution of gas dynamics equations with respect to the group $H(\alpha, \beta)$:

$$\begin{gathered} u = \frac{\beta x - c_1}{\alpha + \beta t}, \quad v = \frac{ty + z - c_3(t \cos \Phi + \sin \Phi)}{t^2 + 1}, \quad w = \frac{tz - y - c_3(t \sin \Phi - \cos \Phi)}{t^2 + 1}, \\ \Phi = \begin{cases} \dfrac{x - (\alpha + \beta t)c_2 - c_1/\beta}{\alpha + \beta t}, & \beta \neq 0, \\ \dfrac{x + (c_1/\alpha)\, t - c_2}{\alpha}, & \beta = 0, \end{cases} \\ \rho = \frac{\rho_0}{(t^2 + 1)(\alpha + \beta t)}, \quad S = \text{const}. \end{gathered} \tag{3.3}$$

The integration constants c_i ($i = 1$, 2, and 3) are arbitrary. Generally speaking, the solution is sought up to transformations admitted by the basic equations (1.1). Therefore, to simplify (3.3), one can use the above transformations from the normalizer $H(\alpha, \beta)$ in G_{11}. Thus, any solution of (3.3) is brought to a standard form, in which c_1 and c_2 are equal to zero. Indeed, these constants are "annihilated" by shear transformations and Galilean translation along the X axis. Moreover, considering solution (3.3) in the class of barochronic motions, for $c_3 \neq 0$ and $\alpha \neq 0$, one can obtain $c_3 = 1$ and $\alpha = 1$ by dilatation transformation $(u, x) \to \alpha(u, x)$,

$(v, y, w, z) \to c_3(v, y, w, z)$ [one can easily see that it is admitted both by the equations of barotropic gas motions and by the factor-system (3.2)]. The equivalent form of solution (3.3) for $c_3 \neq 0$ and $\alpha \neq 0$ is as follows:

$$u = \frac{\beta x}{1+\beta t}, \quad v = \frac{ty + z - (t\cos\Phi + \sin\Phi)}{t^2+1}, \quad w = \frac{tz - y - (t\sin\Phi - \cos\Phi)}{t^2+1},$$
$$\Phi = \frac{x}{1+\beta t}, \quad \rho = \frac{\rho_0}{(t^2+1)(1+\beta t)}, \quad S = \text{const.} \tag{3.4}$$

4. Particle Trajectories. The particle trajectories in solution (3.4) are described by the formulas

$$x = x_0(1+\beta t), \quad y = y_0 + tz_0 - t\sin x_0, \quad z = z_0 - ty_0 + t\cos x_0. \tag{4.1}$$

One can see that the particle trajectories are straight lines. In this case, the particle motion as a whole is nontrivial. Let, for example, for $t = 0$, the particles be on a straight line parallel to the X axis. From (4.1) it is obvious that at $t > 0$, they form a spiral with period $2\pi(1+\beta t)$ and radius t. Remaining parallel to the X axis, the axis of the spiral sweeps out the plane. One of the peculiarities of the solution is that, if $\beta < 0$, at time $t = -1/\beta$ a collapse occurs: the spiral "sticks together" in a circle, while the density increases to infinity.

5. Characteristic Conoid. On solution (3.4), the sound characteristics of the gas dynamics equations are found in the form $h(t, \mathbf{x}) = \text{const}$. The corresponding equations are as follows:

$$h_t + uh_x + vh_y + wh_z = \pm c\sqrt{h_x^2 + h_y^2 + h_z^2}. \tag{5.1}$$

For Eqs. (5.1), in the case of characteristics C_+, the equations of bicharacteristics have the form

$$d\mathbf{x}/dt = \mathbf{u} + c\nabla h/|\nabla h|, \quad dh_j/dt = -\mathbf{u}_j \cdot \nabla h - c_j|\nabla h| \quad (j = t, x, y, z), \tag{5.2}$$

where c is the sound velocity, and the subscript j denotes the derivative with respect to the corresponding arguments. *A characteristic conoid* is a geometrical place of all bicharacteristics (5.2) going out from the given point $P(\mathbf{x}_0, t_0)$.

We take the equation of state of a polytropic gas $p = \rho^\gamma$. For simplicity, we set $c_3 = 0$ in solution (3.3). Then, the sound velocity c is $c = \sqrt{p_\rho} = \sqrt{\gamma}(1/[(t^2+1)(1+\beta t)])^{(\gamma-1)/2}$.

Integration of system (5.2) yields the relations

$$x = x_0 + \beta t x_0 + (1+\beta t)\int_0^t \frac{Q(t)}{(1+\beta t)^2}\,dt, \quad y = y_0 + \frac{s}{r}ty_0 + (r+st)\int_0^t \frac{Q(t)}{t^2+1}\,dt,$$
$$z = z_0 - \frac{r}{s}tz_0 + (-rt+s)\int_0^t \frac{Q(t)}{t^2+1}\,dt, \quad Q(t) = c(t)\Big(\frac{r^2+s^2}{t^2+1} + \frac{1}{(1+\beta t)^2}\Big)^{-1/2}. \tag{5.3}$$

Equations (5.3) with r and s, which take on all real values, yield the parametric form of the characteristic conoid.

6. The Case $\alpha^2 + \beta^2 = 0$. In this case, as was shown in Sec. 2, a nonsingular invariant H-solution cannot be constructed. Consider a partially invariant solution with respect to the subgroup $H(0,0)$. The universal invariant of the group looks as follows:

$$J = (t, x, u, U, S, \rho). \tag{6.1}$$

Here the function U is determined from the relations

$$U\cos\Phi = v - (ty+z)/(t^2+1), \quad U\sin\Phi = w - (tz-y)/(t^2+1). \tag{6.2}$$

According to [3], the existence of a partially invariant H-solution of rank $\sigma < n$ requires that the inequalities $1 \leqslant \delta \leqslant 2$ hold for defect δ. Thus, there are two possibilities: a solution of defect $\delta = 1$ and rank $\sigma = 2$ and also a solution of defect $\delta = 2$ and rank $\sigma = 3$. Below, we study the first possibility.

The solution is obtained if we seek $m - \delta = 4$ from invariants (6.1) as functions of the other two, i.e., the invariant functions u, U, ρ, and S depend in this solution only on t and x. Generally speaking, the remaining, "superfluous," function Φ depends on all independent variables (t, x, y, and z).

Having preliminarily expressed v and w from (6.2), we substitute the solutions obtained for the unknown functions in (1.1). Using the auxiliary function $h(t,x)$, we write the substitution results. The system splits into two subsystems: the invariant subsystem

$$u_t + uu_x + \frac{1}{\rho}p_x = 0, \quad \frac{1}{\rho}(\rho_t + u\rho_x + \rho u_x) + \frac{2t}{t^2+1} - Uh = 0,$$
$$S_t + uS_x = 0, \quad U_t + uU_x + \frac{Ut}{t^2+1} = 0 \tag{6.3}$$

and the additional subsystem for the "superfluous" function Φ:

$$\Phi_t + u\Phi_x + \left(U\cos\Phi + \frac{ty+z}{t^2+1}\right)\Phi_y + \left(U\sin\Phi + \frac{tz-y}{t^2+1}\right)\Phi_z - \frac{1}{t^2+1} = 0,$$
$$\sin\Phi\,\Phi_y - \cos\Phi\,\Phi_z - h = 0. \tag{6.4}$$

At this stage, the question of the existence of a required solution is reduced to studying the compatibility of the overdetermined system (6.4).

Proposition. *System* (6.4) *is compatible if and only if* $h(t,x) \equiv 0$. *In this case, the required partially invariant solution is reduced to invariant.*

To reduce system (6.4) to involution, it is convenient to find the dependence $\Phi = \Phi(t,x,y,z)$ in implicit form: $F(t,x,y,z,\Phi) = 0$. Then, Eqs. (6.4) are representable as the action of linear operators on F:

$$\Omega_1 = \partial_t + u\partial_x + \left(U\cos\Phi + \frac{ty+z}{t^2+1}\right)\partial_y + \left(U\sin\Phi + \frac{tz-y}{t^2+1}\right)\partial_z + \frac{1}{t^2+1}\partial_\Phi,$$
$$\Omega_2 = \sin\Phi\partial_y - \cos\Phi\partial_z + h\partial_\Phi.$$

Thus, (6.4) is equivalent to the system

$$\Omega_1 F = 0, \quad \Omega_2 F = 0. \tag{6.5}$$

Generally speaking, system (6.5) is *active*, i.e., it can produce new independent equations. For operators Ω_1 and Ω_2, we form a *commutator* $[\Omega_1,\Omega_2] = \Omega_1\Omega_2 - \Omega_2\Omega_1$. Denote

$$\Omega_3 = [\Omega_1,\Omega_2] + \frac{1}{t^2+1}\Omega_2 = \left[\frac{2\cos\Phi}{t^2+1} + hU\sin\Phi\right]\partial_y + \left[\frac{2\sin\Phi}{t^2+1} - hU\cos\Phi\right]\partial_z + \left[h_t + uh_x + \frac{ht}{t^2+1}\right]\partial_\Phi.$$

Clearly, the function F should also be an invariant of the operator Ω_3. At the same time, the operator Ω_3 is linearly independent (the linear combination can be taken with coefficients depending on all independent variables t, x, y, z, and Φ) with the operators Ω_1 and Ω_2, since the consequence of the expression $\Omega_3 = \lambda^1\Omega_1 + \lambda^2\Omega_2$ is a contradictory equality $2/(t^2+1) = 0$. Thus, one should join the equation $\Omega_3 F = 0$ to system (6.5).

We calculate the operator Ω_4 in a similar way:

$$\Omega_4 = [\Omega_2,\Omega_3] = \left[\left(h^2U - h_t - uh_x - \frac{ht}{t^2+1}\right)\cos\Phi - h\,\frac{2\sin\Phi}{t^2+1}\right]\partial_y$$
$$+ \left[\left(h^2U - h_t - uh_x - \frac{ht}{t^2+1}\right)\sin\Phi + h\,\frac{2\cos\Phi}{t^2+1}\right]\partial_z.$$

For the linear dependence of Ω_4 on Ω_1, Ω_2, and Ω_3, it is necessary that the equation

$$\frac{2h^2}{t^2+1} + \frac{t^2+1}{2}\left(h^2U - h_t - uh_x - \frac{ht}{t^2+1}\right)^2 = 0$$

be satisfied for the function h, from which follows

$$h(t,x) \equiv 0. \tag{6.6}$$

It is clear that otherwise, one can express the operator ∂_Φ by Ω_2, Ω_3, and Ω_4 (by a linear combination of them), i.e., the equation $F_\Phi = 0$ takes places, which implies the incompatibility of system (6.4). In the case

(6.6), the expressions for operators Ω_i $(i = 1,\ 2,\ 3)$ are simplified:

$$\Omega_1 = \partial_t + u\partial_x + \frac{ty+z}{t^2+1}\partial_y + \frac{tz-y}{t^2+1}\partial_z + \frac{1}{t^2+1}\partial_\Phi,$$
$$\Omega_2 = \sin\Phi\,\partial_y - \cos\Phi\,\partial_z, \quad \Omega_3 = \cos\Phi\,\partial_y + \sin\Phi\,\partial_z.$$

One can easily verify that this system of operators is in involution. Turning back to the explicit definition of the function Φ, we obtain the system of equations in involution which is equivalent to (6.4): $\Phi_t + u\Phi_x = 1/(t^2+1)$ and $\Phi_y = 0$, $\Phi_z = 0$.

Note that the dependence of the "superfluous" function Φ only on the invariant variables t and x follows from the latter two equations. This indicates that the solution is actually reduced to invariant. Indeed, the desired solution is invariant with respect to the subgroup generated by the operators H_1 and H_2.

For $h \equiv 0$, factor-system (6.3) is simplified and, together with the equations for Φ, takes the form

$$\begin{gathered} u_t + uu_x + p_x/\rho = 0, \quad \rho_t + u\rho_x + \rho(u_x + 2t/(t^2+1)) = 0, \quad S_t + uS_x = 0, \\ U_t + uU_x + Ut/(t^2+1) = 0, \quad \Phi_t + u\Phi_x = 1/(t^2+1), \quad \Phi_y = 0, \quad \Phi_z = 0, \quad p = f(\rho, S). \end{gathered} \tag{6.7}$$

7. Integration of System (6.7). We replace the variables $R = \rho(t^2+1)$ and write the first two equations of (6.7) as

$$u_t + uu_x + (t^2+1)p_x/R = 0, \quad R_t + (uR)_x = 0. \tag{7.1}$$

The Lagrangian coordinate $\xi(t,x)$ is introduced by the relations $R = \xi_x$ and $Ru = -\xi_t$. In view of these relations, the second equation of (7.1) holds automatically. The last equations of (6.7) are integrated: $S = S(\xi)$, $U = U_0(\xi)/\sqrt{t^2+1}$, and $\Phi = \arctan t + \Phi_0(\xi)$. Here $S(\xi)$, $U_0(\xi)$, and $\Phi_0(\xi)$ are arbitrary functions of their argument. Calculating the derivatives and substituting them into the first equation of (7.1), we obtain the equation for the Lagrangian coordinate ξ:

$$\xi_x^2\xi_{tt} - 2\xi_t\xi_x\xi_{tx} + (\xi_t^2 - c^2\xi_x^2)\,\xi_{xx} = (t^2+1)\,\xi_x^3 f_S S'(\xi), \tag{7.2}$$

where $c^2 = f_\rho(\rho, S)$ is the squared sound velocity. Thus, system (6.7) reduces to one quasi-linear differential equation in second-order partial derivatives (7.2).

One can study factor-system (6.7) using the methods of group analysis of differential equations. We note that the first three equations of system (6.7) form an independent subsystem for the functions u, S, and ρ. Knowledge of these functions allows one to integrate easily the other equations. Moreover, in group analysis, it is more convenient to use (instead of the equation of entropy) the equation for pressure $Dp + A(p,\rho)\operatorname{div}\mathbf{u} = 0$, where $A(p,\rho)$ is a prescribed function of state [its physical meaning is $A(p,\rho) = \rho c^2$]. It is clear that the function ρ will be an invariant of the H group, i.e., to construct the H-solution, one should assume that p depends on the invariant variables t and x. Below, we study the following system:

$$u_t + uu_x + p_x/\rho = 0, \quad \rho_t + u\rho_x + \rho(u_x + 2t/(t^2+1)) = 0, \quad p_t + up_x + A(p,\rho)(u_x + 2t/(t^2+1)) = 0. \tag{7.3}$$

8. Group Classification. For system (7.3), we solve the problem of group classification with respect to an "arbitrary element," the function $A(p,\rho)$. The required operators are written in the form

$$X = \xi^t\partial_t + \xi^x\partial_x + \xi^u\partial_u + \xi^p\partial_p + \xi^\rho\partial_\rho.$$

Verification of the known criterion from [5] shows that system (7.3) is x-autonomous for an arbitrary function $A(p,\rho)$, i.e., the coordinates ξ^t and ξ^x can depend only on the variables t and x. With allowance for this fact, the system of governing equations is reduced to the following. The coordinate ξ^t is independent of x, i.e., it can depend only on t. For the coordinates ξ^u, ξ^p, and ξ^ρ, we have the expressions

$$\xi^u = \xi_t{}^x - u\xi_t^t + u\xi_x^x, \quad \xi^p = \varphi(t)\,p + \psi(t), \quad \xi^\rho = \rho(\varphi(t) + 2\xi_t^t - 2\xi_x^x), \tag{8.1}$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions of the variable t. The coordinates ξ^t and ξ^x should satisfy the relations

$$2\xi_{tx}^x = \xi_{tt}^t, \quad \xi_{xx}^x = \xi_{tt}^x = 0, \quad \varphi_t + 3\xi_{tx}^x + 2t\xi_t{}^t/(t^2+1) + 2(1-t^2)\,\xi^t/(t^2+1)^2 = 0. \tag{8.2}$$

The latter equation is used to determine $\varphi(t)$, while the other yields the representation for ξ^t and ξ^x

$$\xi^x = c_0 tx + c_1 t + c_2 x + c_3, \quad \xi^t = c_0 t^2 + b_1 t + b_2 \tag{8.3}$$

with constants c_i $(i = 0, \ldots, 3)$ and b_j $(j = 1, 2)$.

Classifying are the equations

$$\begin{gathered} A\varphi = (\varphi p + \psi)\, A_p + \rho(\varphi + 2\xi^t_t - 2\xi^x_x)\, A_\rho, \quad KA(p,\rho) = Rp - \psi_t, \\ K = \xi^x_{tx} + \frac{2t}{t^2+1}\,\xi^t_t + \frac{2(1-t^2)}{(t^2+1)^2}\,\xi^t, \quad R = K + 2\xi^x_{tx}. \end{gathered} \tag{8.4}$$

In view of the dependence of K, R, and ψ only on t, from the second equation of (8.4) we obtain

$$KA_\rho = 0. \tag{8.5}$$

We consider the case $K = 0$. From (8.4) it follows that $\psi_t = 0$, $\xi^x_{tx} = 0$, and $2t\xi^t_t + (2(1-t^2)/(t^2+1))\xi^t = 0$, whence follows that $c_0 = 0$ in (8.3). With allowance for (8.1)–(8.3), we obtain the constants $\varphi = \varphi_0$ and $\psi = \psi_0$ and also

$$\xi^t = 0, \quad \xi^x = c_1 t + c_2 x + c_3, \quad \xi^u = c_1 + c_2 u, \quad \xi^p = \varphi_0 p + \psi_0, \quad \xi^\rho = \rho(\psi_0 - 2c_2). \tag{8.6}$$

We transform the first equation of (8.4):

$$\varphi_0(pA_p + \rho A_\rho - A) + \psi_0 A_p - 2c_2 \rho A_\rho = 0. \tag{8.7}$$

With arbitrary values of A, A_p, and A_ρ, Eq. (8.7) is valid only if $\varphi_0 = 0$, $\psi_0 = 0$, and $c_2 = 0$, i.e., the kernel of the basic groups of Eqs. (7.3) is a two-parameter Lie group with operators

$$X_1 = \partial_x, \quad X_2 = t\partial_x + \partial_u. \tag{8.8}$$

To perform the group classification, we found the group of equivalence transformations. Its factor-group with respect to kernel is generated with respect to the operators $X^a_1 = x\partial_x + u\partial_u - 2\rho\partial_\rho$, $X^a_2 = \partial_p$, and $X^a_3 = p\partial_p + \rho\partial_\rho + A\partial_A$. Thus, the equivalence transformations $A(p,\rho)$ form a three-parameter group acting in the formulas $p' = \alpha_3 p + \alpha_2$, $\rho' = \alpha_1\alpha_3\rho$, and $A' = \alpha_3 A$ (α_i are arbitrary parameters, with $\alpha_1 > 0$ and $\alpha_3 > 0$).

Thus, for $K = 0$, the expression for the operator coordinates is given by formulas (8.6) with the only relation (8.7) relating the constants φ_0, ψ, and c_2. Equation (8.7) was analyzed using the same method as in [3]. As a result, we obtained all cases of dilatation of the kernel of the basic groups listed in Table 1, except the case Nos. 7 and 13.

It remains to consider the second possibility offered by Eq. (8.5): $A_\rho = 0$. Then, from the second equation of (8.4), it follows (up to equivalence transformations) that $A = \gamma p$, $\gamma = \text{const}$, and

$$(\gamma - 3)\,\xi^x_{tx} + (\gamma - 1)\Big(\frac{2t}{t^2+1}\,\xi^t_t + \frac{2(1-t^2)}{(t^2+1)^2}\,\xi^t\Big) = 0, \quad \psi_t = 0. \tag{8.9}$$

New dilatations of the kernel (8.8) can be obtained only for $\gamma = 1$ and $5/3$. In both cases, from the first equation of (8.4), $\psi \equiv 0$. For $\gamma = 5/3$, substituting (8.3) in (8.9), we find $c_0 = b_2$ and $c_1 = 0$. Using (8.1) and (8.2), we obtain

$$\begin{gathered} \xi^t = c_0(t^2+1), \quad \xi^x = c_0 tx + c_1 t + c_2 x + c_3, \\ \xi^u = c_0 x + c_1 + u(c_2 - c_0 t), \quad \xi^p = (-5c_0 t + \varphi_0)p, \quad \xi^\rho = \rho(-3c_0 t - 2c_2 + \varphi_0). \end{gathered}$$

The kernel of the basic groups dilates into three operators: Y_1, Y_2, and Y_6.

With $\gamma = 1$, from (8.9) $\xi^x_{tx} = 0$, i.e., $c_0 = 0$ in (8.3). Integrating the last equation of (8.2), we find the expression $\varphi(b_1, b_2) = -2b_1 t^2/(t^2+1) - 2b_2 t/(t^2+1) + \varphi_0$ whose substitution in (8.1) yields $\xi^t = b_1 t + b_2$, $\xi^x = c_1 t + c_2 x + c_3$, $\xi^u = c_1 - u(b_1 - c_2)$, $\xi^p = \varphi(b_1, b_2)p$, and $\xi^\rho = \rho(\varphi(b_1, b_2) + 2b_1 - 2c_2)$ for the coordinates of the operator X. The kernel of basic groups dilates into four operators: Y_1, Y_2, Y_4, and Y_5.

TABLE 1

Number of the kernel dilatation	A	k	Y
1	$f(p,\rho)$	2	—
2	$pf(p\rho^{-\gamma})$, $\gamma \neq 0{,}1$	3	$(\gamma-1)Y_1+2\gamma Y_2$
3	$pf(p/\rho)$	3	Y_2
4	$f(p)$	3	Y_1
5	$pf(\rho)$	3	Y_1+Y_2
6	γp, $\gamma \neq 0;\ 1;\ 5/3$	4	Y_1, Y_2
7	$(5/3)p$	5	Y_1, Y_2, Y_6
8	$f(\rho e^{-p})$	3	$-Y_1+2Y_3$
9	$f(\rho)$	3	Y_3
10	$\gamma\rho^\gamma$, $\gamma \neq 0.1$	4	$(\gamma-1)Y_1+2\gamma Y_2, Y_3$
11	ρ	4	Y_2, Y_3
12	1	4	Y_1, Y_3
13	p	6	Y_1, Y_2, Y_4, Y_5
14	0	∞	Y_1, Y_φ

The results of group classification are presented in Table 1. One can find there the form of the function $A(p,\rho)$ specifying the given dilatation of the kernel, the dimension of the admitted transformation group, and the operators dilating the kernel of the basic groups (8.8). The following operators are involved in dilatation of the kernel:

$$Y_1 = x\partial_x + u\partial_u - 2\rho\partial_\rho, \quad Y_2 = p\partial_p + \rho\partial_\rho, \quad Y_3 = \partial_p,$$

$$Y_4 = \partial_t - \frac{2t}{t^2+1}p\partial_p - \frac{2t}{t^2+1}\rho\partial_\rho, \quad Y_5 = t\partial_t - u\partial_u - \frac{2t^2}{t^2+1}p\partial_p + \frac{2}{t^2+1}\rho\partial_\rho,$$

$$Y_6 = (t^2+1)\partial_t + tx\partial_x + (x-ut)\partial_u - 5tp\partial_p - 3t\rho\partial_\rho, \quad Y_\varphi = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p.$$

Here $\varphi(p)$ is an arbitrary function.

On the basis of this group classification, system (7.3) can be studied more completely with the use of the symmetry properties incorporated in these equations.

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